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An Adaptive Fourth-order Model for Multiplicative Noise Removal

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Abstract

In this paper, we focus on the problem of removing multiplicative (speckle) noise. Furthermore, a fourth-order model has been proposed. We show numerical experiments to illustrate the effectiveness of the suggested method in denoising images contaminated by multiplicative Gamma noise. Our results have been compared with the AA model and the SO model. The results determined the superiority of our model over the two mentioned models.

Keywords: *Fourth-order PDE, Multiplicative Noise ,Noise removal, Finite Difference Method.*

Introduction

Many special purpose image acquisition systems such as Ultrasound imaging, Synthetic Aperture Radar (SAR), Sonar and Laser imaging produce images which are corrupted by datadependent noise. Due to the coherent nature of these image acquisition processes, the standard additive noise model is inadequate for processing such images. Instead, multiplicative noise models provide a better description of coherent imaging systems.

Assume a noise image satisfies

$$u_0 = u\eta \ , \tag{1}$$

where u is the ideal image, u_0 is the noisy image and η is the noise. The problem is then to recover u from u_0. From (1), it is clear thatalmost all the information of the original image may disappear when it is distorted by multiplicative noise, then it's more difficult to remove multiplicative noise than additive noise. In many applications the mean and variance of noise are priori assumed known to satisfy the following conditions

$$\frac{1}{|\Omega|}\int_{\Omega} \eta dx, \quad (2)$$

and

$$\frac{1}{|\Omega|}\int_{\Omega} (\eta-1)^2 dx = \sigma^2, \quad (3)$$

Which follows a Gamma Law with mean one and density function

$$g(\eta) = \frac{L^L}{\Gamma(L)}\eta^{L-1}\exp(-L\eta)\, 1_{\{\eta\geq 0\}}. \quad (4)$$

Here L is the number of looks, $\Gamma(\cdot)$ is the usual gamma function and $1_{\{\eta\geq 0\}}$ is the indicator function of the subset $\{\eta \geq 0\}$.

Many methods have been proposed to solve denoising problem, variational approaches [2,4,8,9,13,17,20], diffusion approaches [3,10,11,14,18,21], and wavelet approaches [1,6,12,15]. Most of literatures deal with the multiplicative noise by TV regularization approach. The first TV-based approach to remove multiplicative Gaussian noise was introduced by Rudin et.al.[20] in 2003. TV regularization was then used for multiplicative Poisson noise removal in [7]. Aubert and Aujol in [2] have derived a multiplicative noise model by the maximum a posteriori (MAP) estimation approach to remove multiplicative Gamma noise. The model is given as

$$u = arg\min_{u\in S(\Omega)}\left\{\int_{\Omega} |Du|dx + \lambda\int_{\Omega} H(u,f)dx\right\}, \quad (5)$$

where$S(\Omega) = \{u > 0, u \epsilon BV(\Omega)\}$. A presentation of proof of the existence and uniqueness of solution of the minimization problem, and indeed that of the corresponding evolution system has been offered by the authors.

Diffusion techniques have been broadly used for image denoising since Perona and Malik [19] developed their nonlinear filter for removing additive noise, in 1990. You and Acton [23], in 2002, proposed a nonlinear diffusion equation for multiplicative noise removal, which tends to produce dislocated and unsharp edges despite of impressive noise reduction performance. Krissian et. al [14] then extend the filter to a matrix anisotropic diffusion, allowing different level of filtering across the image contours and in the principal curvature direction. In 2015, Guo and Zhou [24] proposed a doubly degenerated diffusion model for multiplicative noise removal. The model is given as

$$\partial_t u = div(a(|\nabla u|, u)\nabla u), \quad on\ \Omega \times (0,T), \quad (6)$$
$$\langle \nabla u, n \rangle = 0, \quad on\ \partial\Omega \times (0,T),$$
$$u(x,0) = f(x), \quad on\ \Omega,$$

where $a(|\nabla u|, u) = \frac{2|u|^\alpha}{M^\alpha + |u|^\alpha}\frac{1}{(1+|\nabla u|^2)^{(1-\beta)/2}}$, $\alpha > 0, 0 < \beta < 1, M = sup_{x\in\Omega} u.$

It is well known from the history of image denoising that the second-order nonlinear methods like TV and PM lead to the formation of constant patches during the PDE evolution. Therefore, the filtered output appears blocky or staircased. Different methods were introduced in recent years for reducing the staircase effect associated with second-order PDEs [3,4,16,17].

1. The Proposed Model

In this section, inspired by the impressive performance of the nonlinear diffusion model, we propose a new fourth-order PDE model for multiplicative Gamma noise removal. Because the second-order PDEs have the staircase effect and the fourth-order PDEs can alleviate this effect, a model based on the fourth-order PDEs is suggested for multiplicative noise removal. The proposed model is given as follows

$$u_t = D_{ii}^2\left(\frac{a(x)D_{ii}^2 u}{|D_{ii}^2 u|}\right) = 0, \quad on\ \Omega \times (0,T),$$
$$u(x,t) = 0, \quad on\ \partial\Omega \times (0,T), \quad (7)$$
$$\frac{\partial u}{\partial \vec{n}} = 0\ on\ \partial\Omega \times (0,T),$$

$u(x,0) = u_0(x), \quad on\ \Omega,$

where $a(x) = \left(\frac{G_\sigma * u_0}{M}\right)\frac{1}{1+k|\nabla u_0|^2}$. $p.k > 0, M = sup_{x\in\Omega}(G_\sigma * u_0)(x),$

$G_\sigma = \frac{1}{4\pi\sigma}\exp\left(\frac{-|x|^2}{4\sigma^2}\right), \sigma > 0, \Omega$ is a bounded domain of R^2 with appropriate smooth boundary, T>0 is fixed, $\vec{n}$ denotes the unit outward normal to the boundary $\partial\Omega$.

In the proposed model, we use the a regularization method similar to [8] and [5] under the frame work of [24]. The introduced Gaussian kernel function in the diffusion coefficient bring a lot of advantages in the proposed model. It helps diffusion coefficient to detect the edges more accurately and further provides the model a more efficient way in removing multiplicative noise. The function $a(x)$ is introduced in the model as a gray level indicator. At the low gray level $(a(x) \to 0)$, the diffusion is slow, at higher gray level $(a(x) \to 1)$ the speed of diffusion is fast so that some small features at low gray level are much less smoothed and therefore are preserved.

2. **Numerical Scheme**:

In this part, we present the traditional Finite Difference Method (FDM) of our mode, (7). Assuming $\tau$to be the time step size and h the space grid size, we discretize time and space as follows:

$$t = n\tau, \qquad n = 0,1,2,3,\dots,.$$

$$x = ih, \qquad i = 0,1,2,\dots,I,$$

$$y = jh, \qquad j = 0,1,2,\dots J,$$

where $Ih \times Jh$ is the size of the original image. Let $u_{i,j}^n$ denote approximations of $u(n\tau, ih, jh)$. We define the discrete approximation:

$$D_x u_{i,j}^n = \frac{u_{i+1,j}^n - u_{i-1,j}^n}{2},$$

$$D_y u_{i,j}^n = \frac{u_{i,j+1}^n - u_{i,j-1}^n}{2},$$

$$\Delta_x u_{i,j}^n = (u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n)/h^2,$$

$$\Delta_y u_{i,j}^n = (u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n)/h^2.$$

The discrete explicit scheme of the problem can be written as

$$u_{i,j}^{n+1} = u_{i,j}^{n} - \tau\left[\Delta_x\left(a_{i,j}\frac{\Delta_x u_{i,j}^n}{\left|\Delta_x u_{i,j}^n\right|_\epsilon}\right) + \Delta_y\left(a_{i,j}\frac{\Delta_y u_{i,j}^n}{\left|\Delta_y u_{i,j}^n\right|_\epsilon}\right)\right],$$

$$a_{i,j} = \left(\frac{(G_\sigma * u_0)_{i,j}}{M}\right)\cdot\frac{1}{1 + k\left[(D_x u_0)_{i,j}^2 + \left(D_y u_0\right)_{i,j}^2\right]},$$

$$u_{i,j}^0 = u_0(ih, jh), \quad 0 \le i \le I, 0 \le j \le J,$$

$$u_{i,0}^n = u_{i,1}^n, \quad u_{0,j}^n = u_{1,j}^n, \; u_{I,j}^n = u_{I-1,j}^n, \quad u_{i,J}^n = u_{i,J-1}^n,$$

$$u_{i,0}^n = 0, \; u_{0,j}^n = 0, \quad u_{I,j}^n = 0, \; u_{i,J}^n = 0.$$

Here the MATLAB function ``conv2" is used to represent the two-dimensional discrete convolution transform of the matrix $u_{i,j}$.

3. Numerical Experiments:

In order to demonstrate the ability of the suggested model in removing Multiplicative Gamma noise, we tested two images, Aerial images (400×400) and Hrd image (512×512), for the three algorithms, which are distorted by multiplicative Gamma noise with mean 1 and $L \in \{1,4,10\}$. The results of the proposed model are compared to AA model [2] and SO model [22]. Table (1) shows the numerical results, for all test images, of our model compared with AA and SO by the peak-signal-to noise ratio (PSNR) and mean absolute deviation/error (MAE).

For fair comparison, the parameters and the stopping criterion of all methods were tweaked mutually to achieve the maximal PSNR or the best MAE. The numerical results for all the test images are tabulated in Table (1). We found that $1 \le p \le 2$ and $0 \le k \le 0.1$ gave the best results across all the experiment of the suggested model.

In order to show the ability of the suggested method in removing Gamma noise we have presented the two images Aerial image , Figures 1,3,5 and Hdr image , Figures 2,4,6. The visual quality and even the numerical values of the restored images demonstrate the effectiveness of suggested model over the other two models. One can easily notice that the homogeneous regions and edges are restored well in the proposed method and the staircase effect is reduced considerably.

Table 1:Compersion of Models

	PSNR			MAE		
L	1	4	10	1	4	10
Hrd image (512×512)						
Ours	**18.32**	**21.63**	**24.35**	**21.35**	**15.29**	**10.70**
AA	18.16	19.60	20.01	22.11	17.87	16.54
SO	14.36	20.75	23.58	36.60	16.04	11.37
Aerial images (400×400)						
Ours	22.15	**25.80**	**27.68**	**12.15**	**9.00**	**7.15**
AA	**22.24**	24.05	24.52	12.50	9.87	9.52
SO	17.33	23.71	26.41	25.31	10.85	7.78

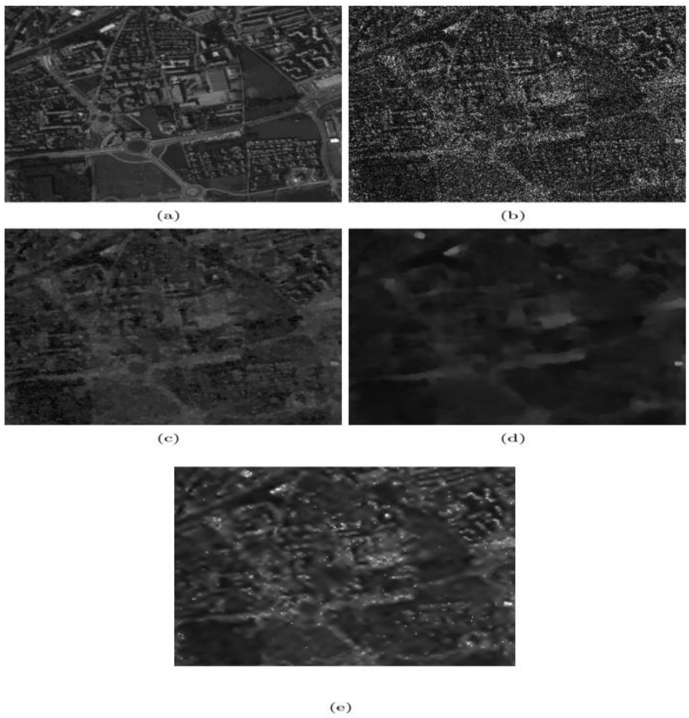

Figure 1: Aerial image (400×400). (a) Original image. (b) Noisy image $L = 1$. (c) AA model. (d) SO model.(e) Our model.

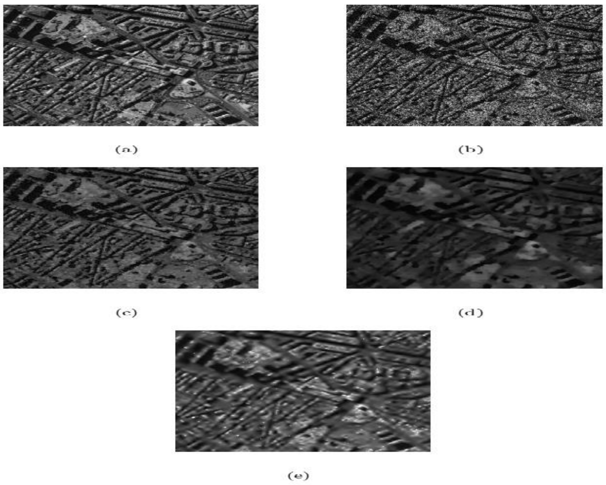

Figure 2: IIrd image (512×512). (a) Original image. (b) Noisy image $L = 1$. (c) AA model. (d) SO model.(e) Our model.

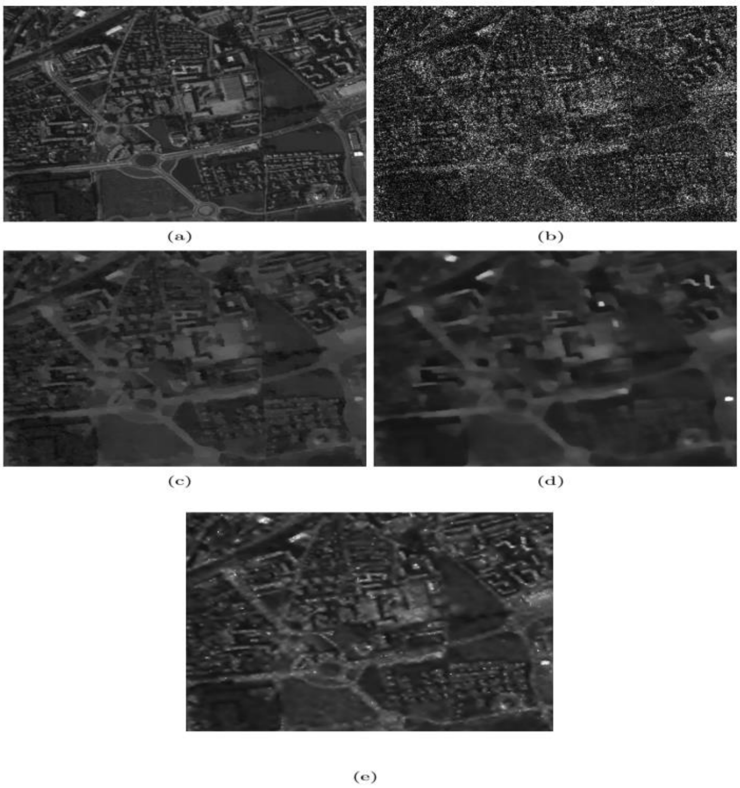

Figure 3: Aerial image (400×400). (a) Original image. (b) Noisy image $L = 4$. (c) AA model. (d) SO model.(e) Our model.

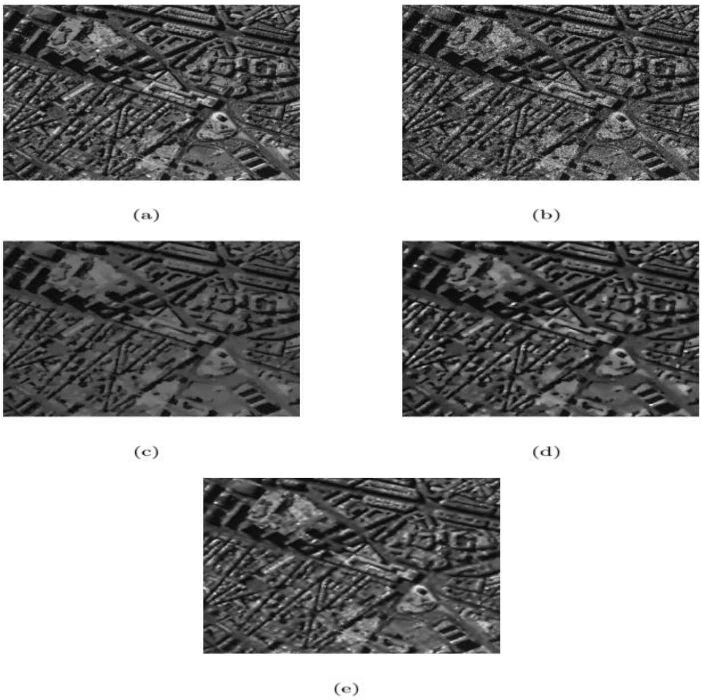

Figure 4: Hdr image (512 × 512). (a) Original image. (b) Noisy image $L = 4$. (c) AA model. (d) SO model.(e) Our model.

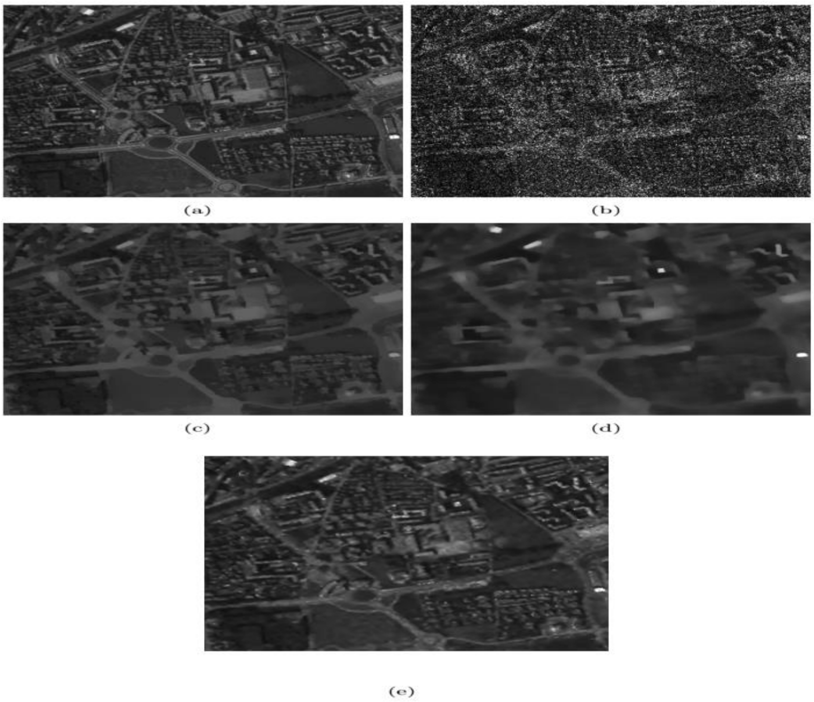

Figure 5: Aerial image (400 × 400). (a) Original image. (b) Noisy image $L = 10$. (c) AA model. (d) SO model.(e) Our model.

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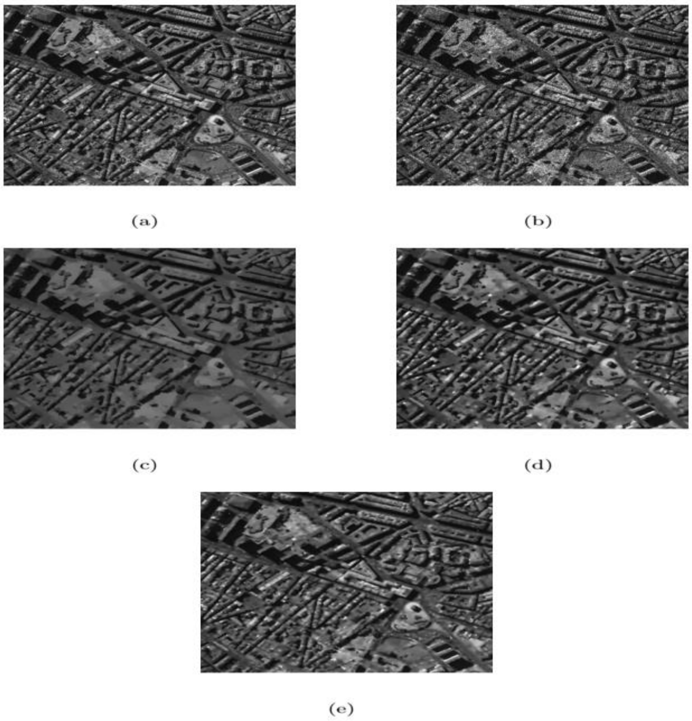

Figure 6: Hdr image (512 × 512). (a) Original image. (b) Noisy image $L = 10$. (c) AA model. (d) SO model.(e) Our model.

Conclusion

In this paper, we proposed a fourth-order model to remove multiplicative noise. Based on the Finite Difference Method (FDM), numerical experiments have been presented to illustrate the effectiveness of the suggested method in image denoising. Numerical results demonstrated the superiority of the suggested model over some famous ones, such as AA model and SO model.

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